

ON NORMAL TENSOR FUNCTORS AND COSET DECOMPOSITIONS FOR FUSION CATEGORIES

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ABSTRACT. In this paper we introduce the notion of double cosets relative to two fusion subcategories of a fusion category. As an application we obtain new criteria to establish if a given tensor functor is normal. We also describe the image of a normal tensor functor between any two fusion categories.

INTRODUCTION

In this paper we introduce the notion of double cosets relative to two fusion subcategories of a given fusion category. The construction of such double cosets is achieved by applying Frobenius-Perron theory for the nonnegative matrices associated to some special operators on the Grothendieck ring of the fusion category.

It is shown that a tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ gives rise to two equivalence relations: one on the set of simple objects $\text{Irr}(\mathcal{C})$ of \mathcal{C} and another on the set $\text{Irr}(\mathcal{D})$ of simple objects of \mathcal{D} . These equivalence relations are reminiscent to the equivalence relations introduced by Rieffel in [10] for the restriction functor attached to an extension of semisimple rings. We prove that the equivalence relation on the set $\text{Irr}(\mathcal{C})$ always arises from a coset decomposition.

In the recent paper [2] the first author and S. Natale have introduced the notion of a normal tensor functor. The kernel of a such tensor functor is called a normal tensor subcategory and can be regarded as a categorification of the notion of a normal subgroup.

A description of images of objects under normal tensor functors $F : \mathcal{C} \rightarrow \mathcal{D}$ is given in Section 4. In Theorem 4.4 it is shown that objects in the same equivalence class of $\text{Irr}(\mathcal{C})$ have similar images.

By analogy to ring theory, we introduce the notion of radical of a fusion subcategory. For a normal fusion subcategory we show that this radical coincides to the commutator of the subcategory. Recall that

the commutator of a fusion subcategory was defined in [8] in order to study Mueger's centralizer for modular fusion categories.

Recall that for a finite group G a G -extension of a fusion category \mathcal{D} is a G -graded fusion category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ with $\mathcal{C}_1 = \mathcal{D}$. We say that such an extension is normal if \mathcal{C}_1 is a normal fusion subcategory of \mathcal{C} . In Theorem 5.4 it is shown that each component \mathcal{C}_g of a normal G -extension $\mathcal{C}_1 \subset \mathcal{C}$ is a left coset with respect to the trivial component.

This paper is organized as follows. In the first section we recall few basic facts on fusion categories and their Grothendieck rings that are needed throughout the paper.

In Section 2 we give the definition the notion of double cosets and prove a decomposition into indecomposable bimodule categories. This notion also generalizes the notion of double cosets for Hopf subalgebras given in [3].

In the next section we introduce the two equivalence relations \approx^F and \approx_F attached to a tensor functor F . It is shown in Proposition 3.3 that the equivalence relation \approx^F is in fact a coset equivalence relation.

In Section 4 one gives a description of the image of simple objects under a normal tensor functor. As an application we consider the forgetful functor $\text{Res}_H^G : \mathcal{C}^G \rightarrow \mathcal{C}^H$, where G is a finite group acting by tensor autoequivalences on a fusion category \mathcal{C} and H is a subgroup of G . A description of the simple objects of an equivariantization was given recently in [5]. It is shown in Proposition 4.9 that Res_H^G is a normal tensor functor if and only if H is a normal subgroup of G .

In Section 5 we introduce the new concept of radical of a fusion subcategory. If the fusion subcategory is a normal fusion subcategory we show that this radical coincides to the commutator of the fusion subcategory. We also show in this section that each homogeneous component of a normal G -extension of fusion categories is a left (and right) coset relative to the trivial component.

1. PRELIMINARIES

In this section we recall the basic facts on fusion categories and tensor functors that are needed in this paper.

Through all this paper we work over the ground field \mathbb{C} of complex numbers. Therefore all fusion categories considered here are \mathbb{C} -linear categories. As usually, by a fusion category we mean a \mathbb{C} -linear semisimple rigid tensor category \mathcal{C} with finitely many isomorphism classes of simple objects, finite dimensional spaces of morphisms, and such that the unit object of \mathcal{C} is simple. We refer the reader to [6] for

basics on fusion categories. A fusion subcategory of a given category is a full monoidal replete subcategory which is also fusion category. Recall that the Grothendieck ring $K_0(\mathcal{C})$ of a fusion category \mathcal{C} is the free \mathbb{Z} -module with a basis given by the isomorphism classes of simple objects of \mathcal{C} . By a virtual object of \mathcal{C} we mean any element of the ring $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$, i.e a complex linear combination of the isomorphism classes of simple object of \mathcal{C} .

The usual duality on the category \mathcal{C} induces an involution $''^*$ on $K_0(\mathcal{C})$ given by $[X]^* := [X^*]$. Denote by $\Lambda_{\mathcal{C}}$ the set of simple objects of \mathcal{C} . For a simple object X let $\text{FPdim}(X)$ denote the Frobenius-Perron dimension of X . Recall that $\text{FPdim}(X)$ is the Frobenius-Perron eigenvalue of the operator $L_{[X]}$ of left multiplication by $[X]$ on the Grothendieck ring $K_0(\mathcal{C})$.

Let $R_{\mathcal{C}}$ be the virtual regular object of \mathcal{C} given by:

$$R_{\mathcal{C}} = \sum_{X \in \Lambda_{\mathcal{C}}} \text{FPdim}(X)[X]$$

Note that $[X]R_{\mathcal{C}} = R_{\mathcal{C}}[X] = \text{FPdim}(X)R_{\mathcal{C}}$ for any element X of the Grothendieck ring $K_0(\mathcal{C})$. (see [6]). In particular $R_{\mathcal{C}}^2 = \text{FPdim}(\mathcal{C})R_{\mathcal{C}}$ where the Frobenius-Perron dimension of \mathcal{C} is defined as $\text{FPdim}(\mathcal{C}) := \sum_{X \in \Lambda_{\mathcal{C}}} \text{FPdim}(X)^2$ is the Frobenius-Perron dimension of \mathcal{C} .

There is a bilinear form $m : K_0(\mathcal{C}) \otimes K_0(\mathcal{C}) \rightarrow k$ defined as follows: if X and Y are two objects of \mathcal{C} then $m_{\mathcal{C}}([X], [Y]) = \dim_k \text{Hom}_{\mathcal{C}}(X, Y)$. The following properties of m will be used later in the paper:

$$(1.1) \quad m_{\mathcal{C}}([X], [Y][Z]) = m_{\mathcal{C}}([Y]^*, [Z][X]^*) = m_{\mathcal{C}}([Z]^*, [X]^*[Y]) \text{ and}$$

$$(1.2) \quad m_{\mathcal{C}}([X], [Y]) = m_{\mathcal{C}}([Y], [X]) = m_{\mathcal{C}}([Y]^*, [X]^*)$$

for all $[X], [Y], [Z] \in K_0(\mathcal{C})$.

Recall that a tensor functor between two fusion categories \mathcal{C} and \mathcal{D} is a pair $(F, J) : \mathcal{C} \rightarrow \mathcal{D}$ of a functor F with $F(1) \cong 1$ and a natural transformation J such that for any two objects $X, Y \in \mathcal{C}$ the morphisms $J_{X,Y} : F(X \otimes Y) \xrightarrow{\cong} F(X) \otimes F(Y)$ are natural isomorphisms in \mathcal{D} . Moreover $J_{X,Y}$ satisfy some additional natural compatibility conditions.

2. COSET DECOMPOSITION FOR FUSION CATEGORIES

2.1. Double coset decomposition for fusion categories. Let \mathcal{C} be a fusion category and \mathcal{D}, \mathcal{E} be two fusion subcategories of \mathcal{C} . Define an equivalence relation $r_{\mathcal{D}, \mathcal{E}}^{\mathcal{C}}$ on the set of simple objects $\Lambda_{\mathcal{C}}$ as follows: $X \sim Y$ if there are objects $D \in \Lambda_{\mathcal{D}}$ and $E \in \Lambda_{\mathcal{E}}$ such that Y is a constituent of $D \otimes X \otimes E$. It can be easily checked that $X \sim Y$ if

and only if $m_{\mathcal{C}}([X], R_{\mathcal{D}}[Y]R_{\mathcal{E}}) > 0$ where $R_{\mathcal{D}}$ and $R_{\mathcal{E}}$ are the regular virtual objects of \mathcal{D} and \mathcal{E} .

It is not difficult to check that $r_{\mathcal{D}, \mathcal{E}}^{\mathcal{C}}$ is an equivalence relation. Clearly $X \sim X$ for any $X \in K_0(\mathcal{C})$ since both $R_{\mathcal{C}}$ and $R_{\mathcal{D}}$ contain the trivial character.

Using the above properties of the bilinear form m , one can see that if $X \sim Y$ then

$$\begin{aligned} m([Y], R_{\mathcal{C}}[X]R_{\mathcal{D}}) &= m(R_{\mathcal{C}}^*, [X]R_{\mathcal{D}}[Y]^*) = m([X]^*, R_{\mathcal{D}}[Y]^*R_{\mathcal{C}}) \\ &= m([X], R_{\mathcal{C}}^*[Y]R_{\mathcal{D}}^*) = m([X], R_{\mathcal{C}}[Y]R_{\mathcal{D}}) \end{aligned}$$

since $R_{\mathcal{C}}^* = R_{\mathcal{D}}$ and $R_{\mathcal{D}}^* = R_{\mathcal{C}}$. Thus $Y \sim X$.

To check the transitivity suppose that $X \sim Y$ and $Y \sim Z$. Then X is a subobject of $R_{\mathcal{D}}(R_{\mathcal{D}}ZR_{\mathcal{E}})R_{\mathcal{E}} = \text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})R_{\mathcal{D}}ZR_{\mathcal{E}}$ and therefore $X \sim Z$.

2.2. Principal eigenvectors. Let $\Lambda_1, \Lambda_2, \dots, \Lambda_l$ be the equivalence classes of the equivalence relation $r_{\mathcal{D}, \mathcal{E}}^{\mathcal{C}}$ on $\Lambda_{\mathcal{C}}$ and define

$$(2.1) \quad A_i = \sum_{X \in \Lambda_i} \text{FPdim}(X)[X]$$

for $1 \leq i \leq l$.

For any element $X \in \Lambda_{\mathcal{C}}$ let L_X and R_X be the linear operators given by the left and right multiplication with $[X]$ on the Grothendieck ring $K_0(\mathcal{C})$.

Proposition 2.1. *With the above notations it follows that A_i are eigenvectors of the operator $T = L_{R_{\mathcal{D}}} \circ R_{R_{\mathcal{E}}}$ on $K_0(\mathcal{C})$ corresponding to the eigenvalue $\text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})$.*

Proof. Definition of $r_{\mathcal{D}, \mathcal{E}}^{\mathcal{C}}$ implies that $X \sim X'$ if and only if one has $m_{\mathcal{C}}(X', T(X)) > 0$. It follows that the objects $T(A_i)$ have all the irreducible constituents in Λ_i for all $1 \leq i \leq l$. Note that $R_{\mathcal{C}} = \sum_{i=1}^l A_i$ and formula

$$(2.2) \quad T(R_{\mathcal{C}}) = \text{FPdim}(\mathcal{E})\text{FPdim}(\mathcal{D})R_{\mathcal{C}}$$

gives that $T(A_i) = \text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})A_i$ for all $1 \leq i \leq l$. \square

In the sequel, we will use the Frobenius-Perron theorem for matrices with nonnegative entries (see [7]). If A is such a matrix then A has a positive eigenvalue λ which has the biggest absolute value among all the other eigenvalues of A . The eigenspace corresponding to λ has a unique vector with all entries positive. λ is called the principal value of A and the corresponding positive vector is called the principal vector of

A . Also the eigenspace of A corresponding to λ is called the principal eigenspace of the matrix A . The following result appears in [7]:

Proposition 2.2. ([7], Proposition 5.) *Let A be a matrix with non-negative entries such that A and A^t have the same principal eigenvalue and the same principal vector. Then after a permutation of the rows and the same permutation of the columns A can be decomposed in diagonal blocks $A = A_1, A_2, \dots, A_l$ with each block an indecomposable matrix.*

Recall also from [7] that a matrix $A \in M_n(\mathbb{C})$ is called decomposable if the set $I = \{1, 2, \dots, n\}$ can be written as a disjoint union $I = J_1 \cup J_2$ such that $a_{uv} = 0$ whenever $u \in J_1$ and $v \in J_2$. Otherwise the matrix A is called indecomposable.

Theorem 2.3. *Let \mathcal{C} be a fusion category and \mathcal{D}, \mathcal{E} be two fusion subcategories of \mathcal{C} . Consider the linear operator $T = L_{R_{\mathcal{D}}} \circ R_{R_{\mathcal{E}}}$ on the Grothendieck ring $K_0(\mathcal{C})$ and let $M := [T]$ be the matrix associated to T with respect to the standard basis of $K_0(\mathcal{C})$ given by the set $\Lambda_{\mathcal{C}}$ of simple objects of \mathcal{C} . Then one has the following:*

- 1) *The principal eigenvalue of $[T]$ is $\text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})$.*
- 2) *The eigenspace corresponding to the eigenvalue $\text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})$ has $(A_i)_{1 \leq i \leq l}$ as k -basis where A_i are defined in Equation 4.1.*

Proof. 1) Let λ be the biggest eigenvalue of T and v the principal eigenvector corresponding to λ . Then $R_{\mathcal{C}} v R_{\mathcal{E}} = \lambda v$. Applying Frobenius-Perron dimension on both sides of this relation it follows that

$$\text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})\text{FPdim}(v) = \lambda \text{FPdim}(v).$$

But $\text{FPdim}(v) > 0$ since v has all positive entries. It follows that $\lambda = \text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})$.

2) It is easy to see that the transpose of the matrix M is also M . To check that let X_1, \dots, X_s be the basis of $K_0(\mathcal{C})$ given by the isomorphism classes of simple objects of \mathcal{C} and suppose that $T(X_i) = \sum_{j=1}^s t_{ij} X_j$ with $t_{ij} \geq 0$. Thus $t_{ij} = m_{\mathcal{C}}(X_j, R_{\mathcal{D}} X_i R_{\mathcal{E}})$ and

$$\begin{aligned} t_{ji} &= m_{\mathcal{C}}(X_i, R_{\mathcal{D}} X_j R_{\mathcal{E}}) = m(R_{\mathcal{D}}^*, X_j R_{\mathcal{E}} X_i^*) \\ &= m_{\mathcal{C}}(X_j^*, R_{\mathcal{E}} X_i^* R_{\mathcal{D}}) = m_{\mathcal{C}}(X_j, R_{\mathcal{D}}^* X_i R_{\mathcal{E}}^*) \\ &= t_{ij} \end{aligned}$$

since $R_{\mathcal{D}}^* = R_{\mathcal{D}}$ and also $R_{\mathcal{E}}^* = R_{\mathcal{E}}$.

Then Proposition 2.2 implies that after a permutation of the rows and the same permutation of the columns the matrix M decomposes in diagonal blocks $M = \{M_1, M_2, \dots, M_s\}$ with each block an indecomposable matrix. This decomposition of M in diagonal blocks gives

a partition $\Lambda_{\mathcal{C}} = \cup_{i=1}^s B_i$ on the set of simple objects of \mathcal{C} , where each B_i corresponds to the rows (or columns) indexing the block M_i . The eigenspace of $[T]$ corresponding to the eigenvalue λ is the sum of the eigenspace of the diagonal blocks M_1, M_2, \dots, M_l corresponding to the same value. But since each M_i is an indecomposable matrix it follows that the eigenspace of M_i corresponding to λ is one dimensional (see [7]). If $\bar{B}_j = \sum_{X \in B_j} \text{FPdim}(X)X$ then as in the proof of Proposition 2.1 it can be seen that \bar{B}_j is an eigenvector of T corresponding to the eigenvalue $\lambda = \text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})$. Thus \bar{B}_j is the unique eigenvector of A_j corresponding to the eigenvalue $\text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})$. Therefore each A_i is a linear combination of these vectors. But if $X \in \mathcal{A}_i$ and $X' \in \mathcal{A}_j$ with $i \neq j$ then $m_{\mathcal{C}}(X', T(X)) = 0$ and the definition of $r_{K,L}^H$ implies that $X \approx X'$. This means that A_i is a scalar multiple of some B_j and this defines a bijective correspondence between the diagonal blocks and the equivalence classes of the relation $r_{\mathcal{D}, \mathcal{E}}^{\mathcal{C}}$. Thus the eigenspace corresponding to the principal eigenvalue $\text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})$ has a k -basis given by A_i with $1 \leq i \leq l$. \square

Corollary 2.4. *Let \mathcal{C} be a fusion category and \mathcal{D}, \mathcal{E} be two fusion subcategories \mathcal{C} . Then there is a decomposition of \mathcal{C} as sum of indecomposable \mathcal{D} - \mathcal{E} bimodule categories*

$$\mathcal{C} = \bigoplus_{i=1}^l \mathcal{B}_i$$

where each \mathcal{B}_i is the full abelian subcategory of \mathcal{C} generated by the simple objects contained in B_i .

Proof. Consider as above the equivalence relation $r_{\mathcal{D}, \mathcal{E}}^{\mathcal{C}}$ relative to the fusion subcategories \mathcal{D} and \mathcal{E} . By the definition of the equivalence relation one has $R_{\mathcal{D}}\mathcal{B}_iR_{\mathcal{E}} \subset \mathcal{B}_i$ which shows that \mathcal{B}_i is a \mathcal{D} - \mathcal{E} bimodule. The fact that \mathcal{B}_i is indecomposable bimodule category is immediate. \square

Each bimodule category \mathcal{B}_i from the above corollary will be called a double coset for \mathcal{C} with respect to the fusion subcategories \mathcal{D} and \mathcal{E} .

Corollary 2.5. *With the above notations, if $X \in \mathcal{C}_i$ then*

$$(2.3) \quad R_{\mathcal{D}}[X]R_{\mathcal{E}} = \frac{\text{FPdim}(\mathcal{D})\text{FPdim}(X)\text{FPdim}(\mathcal{E})}{\text{FPdim}(A_i)}A_i$$

Proof. One has that $R_{\mathcal{D}}[X]R_{\mathcal{E}}$ is an eigenvector of $T = L_{R_{\mathcal{D}}} \circ R_{R_{\mathcal{E}}}$ with the maximal eigenvalue $\text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{E})$. From Theorem 2.3 it follows that $R_{\mathcal{D}}[X]R_{\mathcal{E}}$ is a linear combination of the elements A_j . But

$R_{\mathcal{D}}[X]R_{\mathcal{E}}$ cannot contain any simple object of A_j with $j \neq i$ because all the simple objects entering in the decomposition of the above product are in Λ_i . Thus $R_{\mathcal{D}}[X]R_{\mathcal{E}}$ is a scalar multiple of A_i and formula 2.3 follows by applying Frobenius-Peron dimensions. \square

Example 2.6. Let K and L be two Hopf subalgebras of a semisimple Hopf algebra H . Then K and L are also semisimple Hopf algebras and since the characteristic of the based field is zero it follows they are also cosemisimple. By duality, the usual inclusion $i_L : K \hookrightarrow H$ gives a Hopf algebra projection $i_L^* : H^* \rightarrow K^*$. Thus $\text{Rep}(H^*)$ is a fusion category containing $\mathcal{D} := \text{Rep}(K^*)$. Similarly $\text{Rep}(H^*)$ contains $\mathcal{E} = \text{Rep}(L^*)$. Then it can be easily seen that the equivalence relation $r_{\mathcal{D}, \mathcal{E}}^{\mathcal{C}}$ coincides with the equivalence relation $r_{K, L}^H$ on the set of simple right comodules of H^* introduced in [3].

Consider $\mathcal{D} = \text{Vec}$ in the above construction. The equivalence relation $r_{\text{Vec}, \mathcal{E}}^{\mathcal{C}}$ will be denoted by $r_{\mathcal{E}}^{\mathcal{C}, r}$ and its equivalence classes will be called the right cosets of \mathcal{E} inside \mathcal{D} . Similarly one can define the left cosets relative to \mathcal{D} , via $r_{\mathcal{D}, \text{Vec}}^{\mathcal{C}}$ which will be denoted by $r_{\mathcal{D}}^{\mathcal{C}, l}$.

Example 2.7. Let A and B be two simple objects of \mathcal{C} . Put $\mathcal{D} = \langle A \rangle$ and $\mathcal{L} = \langle B \rangle$ the two fusion subcategories of \mathcal{C} generated by these objects. Then the above equivalence relation $r_{\mathcal{D}, \mathcal{E}}^{\mathcal{C}}$ can be written as follows: $X \sim Y$ if and only if

$$(2.4) \quad m_{\mathcal{C}}([X], [A]^n[Y][B]^m) > 0$$

for some integers m, n . Recall that for a negative integer m the power $[X]^m$ is defined as $[X^*]^m$. The above equivalence class will be denoted by $r_{A, B}^{\mathcal{C}}$. If $A = 1$ denote the above equivalence relation by $r_B^{\mathcal{C}, r}$. Similarly define $r_A^{\mathcal{C}, l}$ when $B = 1$.

Remark 2.8. The theory above allows one to extend the notion of double cosets for Hopf subalgebras from [3] to double cosets for co-quasi Hopf subalgebras of a given co-quasi Hopf algebra.

3. TWO EQUIVALENCE RELATIONS ASSOCIATED TO A TENSOR FUNCTORS

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ a tensor functor between fusion categories. Since fusion categories are finite F has a right adjoint functor R . The following Lemma appears in [1]. For sake of completeness we also present its proof below.

Lemma 3.1. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ a tensor functor between finite tensor categories with a right adjoint R . Then*

$$X \otimes R(Y) \cong R(F(X) \otimes Y)$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

Proof. For any $X \in \mathcal{C}$ one has

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X \otimes R(Y), Z) &= \mathrm{Hom}_{\mathcal{C}}(R(Y), X^* \otimes Z) = \\ &= \mathrm{Hom}_{\mathcal{D}}(Y, F(X^* \otimes Z)) = \mathrm{Hom}_{\mathcal{D}}(Y, F(X^*) \otimes F(Z)) = \\ &= \mathrm{Hom}_{\mathcal{C}}(F(X) \otimes Y, F(Z)) = \mathrm{Hom}_{\mathcal{C}}(R(F(X) \otimes Y), Z) \end{aligned}$$

Yoneda's lemma implies now that

$$X \otimes R(Y) \cong R(F(X) \otimes Y)$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. \square

Note that (F, R) is a Hopf monoidal adjunction using the terminology of [1].

Corollary 3.2. *With the above notations, one has that*

$$R(F(X)) \cong X \otimes R(1)$$

for all $X \in \mathcal{C}$.

Proof. Put $Y = 1$ in the above Theorem. \square

3.1. An equivalence relation on $\Lambda_{\mathcal{C}}$ induced by F . Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between fusion categories with a right adjoint functor R . For $X \in \Lambda_{\mathcal{C}}$ denote by X^F the set of all simple factors of $F(X)$. For $Y \in \Lambda_{\mathcal{D}}$ denote Y_F the set of all simple factors of $R(Y)$.

From the adjunction one has that $X \in Y_F \iff Y \in X^F$. Define a relation \sim^F on $\Lambda_{\mathcal{C}}$ by $X \sim X' \iff X^F \cap X'^F \neq \emptyset$. This is equivalent to $\mathrm{Hom}_{\mathcal{D}}(F(X), F(X')) \neq 0$.

Note that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(F(X), F(X')) &= \mathrm{Hom}_{\mathcal{D}}(1, F(X) \otimes F(X')^*) \\ &= \mathrm{Hom}_{\mathcal{D}}(1, F(X \otimes X'^*)) \\ &= \mathrm{Hom}_{\mathcal{C}}(R(1), X \otimes X'^*) \\ &= \mathrm{Hom}_{\mathcal{C}}(X^*, X'^* \otimes R(1)^*) \\ &= \mathrm{Hom}_{\mathcal{C}}(X, R(1) \otimes X') \end{aligned}$$

Thus $X \sim^F X'$ if and only if

$$(3.1) \quad \dim_k \mathrm{Hom}_{\mathcal{C}}(X, R(1) \otimes X') > 0.$$

Similarly it can be proven that $X \sim^F X'$ if and only if $\dim_k \text{Hom}_{\mathcal{C}}(X, X' \otimes R(1)) > 0$ since $R(1)^* = R(1)$.

In general \sim^F is not an equivalence relation, see Remark 3.4 below for a counterexample. We let \approx^F be its transitive closure. By definition this means $X \approx^F X'$ if and only if there is a sequence of simple objects $X_1, \dots, X_n \in \Lambda_{\mathcal{C}}$ such that $X = X_1 \sim^F X_2 \sim^F \dots \sim^F X_n = X'$.

Proposition 3.3. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ a tensor functor between fusion categories with a right adjoint R . Then \approx^F coincides with the both equivalence relations $r_{R(1)}^{C, r}$ and $r_{R(1)}^{C, l}$.*

Proof. Since $R(1)$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$ (see Proposition 5.1 of [2]) it follows from Formula 2.4 that $r_{R(1)}^{C, r}$ and $r_{R(1)}^{C, l}$ coincide.

Equation 3.1 shows that if $X \sim^F X'$ then $X r_{R(1)}^{C, r} X'$. Thus by transitivity if $X \approx^F X'$ then $X r_{R(1)}^{C, r} X'$. The converse follows from the same equation above. \square

Remark 3.4. Let $S_n \subset S_{n+1}$ be the standard inclusion of the symmetric groups in n and respectively $n+1$ letters. Consider F to be the restriction functor from the category of kS_{n+1} -modules to the category of kS_n -modules. Clearly F is a tensor functor and it follows from Theorem 6.19 of [4] that \approx^F is not an equivalence relation.

3.2. An equivalence relation on $\Lambda_{\mathcal{D}}$ induced by F . One can define an equivalence relation on the set of simple objects $\Lambda_{\mathcal{D}}$ of \mathcal{D} by the formula $Y \sim_F Y'$ if and only if $Y_F \cap Y'_F \neq \emptyset$. This is equivalent with the existence of $Z \in \Lambda_{\mathcal{C}}$ such that both Y, Y' are factors of $F(Z)$. In general \sim_F is not transitive but one can take its transitive closure and obtain an equivalence relation denoted by \approx_F .

Lemma 3.5. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ a tensor functor with a right adjoint R . Then $Y \approx_F Y'$ if and only if $\text{Hom}_{\mathcal{D}}(Y, F(R(1))^n \otimes Y') > 0$.*

Proof. Note that $Y \sim_F Y'$ if and only if $\text{Hom}_{\mathcal{C}}(R(Y), R(Y')) > 0$. On the other hand $\text{Hom}_{\mathcal{C}}(R(Y), R(Y')) = \text{Hom}_{\mathcal{D}}(Y, F(R(Y')))$ = $\text{Hom}_{\mathcal{D}}(Y, R(1) \otimes Y')$. Then it can be iteratively observed that $Y \approx_F Y'$ if and only if $\text{Hom}_{\mathcal{D}}(Y, F(R(1))^n \otimes Y') > 0$. \square

Let $\mathcal{D}_1, \dots, \mathcal{D}_{l'}$ the equivalence classes of \approx_F and

$$(3.2) \quad B_i = \sum_{Y \in \mathcal{D}_i} \text{FPdim}(Y)[Y]$$

for $1 \leq i \leq l'$.

Example 3.6. Consider K a Hopf subalgebra of a semisimple Hopf algebra H and let F be the restriction functor $F : \text{Rep}(H) \rightarrow \text{Rep}(K)$.

Then in the paper [4], the equivalence relation \approx^F was denoted by u_K^H and the equivalence relation \approx_F by d_K^H . As explained in [4] these equivalence relations are similar to the equivalence relations introduced by Rieffel in [10]. They arise from the restriction functor attached to an arbitrary extension of semisimple rings in [10].

Remark 3.7. Suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a dominant tensor functor with a right adjoint R . Then $R(R_{\mathcal{D}}) = R_{\mathcal{C}}$. This can be easily checked using the multiplicity for any simple object of \mathcal{C} in $R(R_{\mathcal{D}})$.

Proposition 3.8. *Suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a dominant tensor functor. Using the above notations it follows that $l = l'$ and after a reindexing of the equivalence classes one has*

$$F(A_i) = [\mathcal{C} : \mathcal{D}]B_i$$

and

$$R(B_i) = [\mathcal{C} : \mathcal{D}]A_i$$

for $1 \leq i \leq l$ where $[\mathcal{C} : \mathcal{D}] = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(\mathcal{D})}$ is the index of \mathcal{D} in \mathcal{C} .

Proof. If $X \sim^F Y$ then clearly all the constituents of $F(X)$ and $F(Y)$ are equivalent under \approx_F . Taking the closure under the transitivity it follows that if $X \approx^F Y$ then all the constituents of $F(X)$ and $F(Y)$ are equivalent under \approx_F . Thus $F(\mathcal{C}_i)$ has all the constituents inside an equivalence class of \approx_F . On the other hand it can easily be seen in the same manner that $F(\mathcal{C}_i)$ cannot be a union of at least two of such equivalence classes.

Since F is dominant it follows from [6] that

$$F(R_{\mathcal{C}}) = [\mathcal{D} : \mathcal{C}]R_{\mathcal{D}}$$

Since $R_{\mathcal{C}} = \sum_{i=1}^l A_i$ and $R_{\mathcal{D}} = \sum_{i=1}^l B_i$ one obtains the first above equality. On the other hand $R(R_{\mathcal{D}}) = R_{\mathcal{C}}$ and one obtains the second equality. \square

Remark 3.9. A similar statement can be written for any functor F , not necessarily dominant. Indeed, one can consider $\mathcal{D}' \subset \mathcal{D}$ the dominant image of F and apply the previous Proposition for the dominant tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}'$.

4. NORMAL TENSOR FUNCTORS

In this section we study the previous equivalence relations for a normal tensor functor F . The definition of a normal tensor functor was given in [2]. If \mathcal{C} and \mathcal{D} are fusion categories then $F : \mathcal{C} \rightarrow \mathcal{D}$ is normal if and only if the following property is satisfied: if $m_{\mathcal{C}}(1, F(X)) > 0$

then $F(X) = \text{FPdim}(X)1$ for any simple object $X \in \Lambda_{\mathcal{C}}$. Then this condition can be rephrased as following:

Lemma 4.1. *The functor F is normal if and only if $\{1_{\mathcal{D}}\}$ is an equivalence class for \approx_F .*

Example 4.2. Consider the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ where $\mathcal{Z}(\mathcal{C})$ is the Drinfeld center of the fusion category \mathcal{C} . Then the class of unit element $1_{\mathcal{C}}$ under the equivalence relation \approx_F is given by $\mathcal{O}(\mathcal{C}_{ad})$, the set of simple objects of the adjoint fusion subcategory of \mathcal{C} . Indeed if R is the right adjoint of F then it follows from Proposition 5.4 [6] that $F(R(1)) = \bigoplus_{X \in \Lambda_{\mathcal{C}}} X \otimes X^*$. Therefore the fusion subcategory generated by $F(R(1))$ is \mathcal{C}_{ad} . Then Lemma 4.1 implies that F is normal if and only if $\mathcal{C}_{ad} = \{1_{\mathcal{C}}\}$.

Recall that \ker_F is the fusion subcategory of \mathcal{C} generated by all simple objects $X \in \Lambda_{\mathcal{C}}$ such that $F(X) = \text{FPdim}(X)1$.

Theorem 4.3. *Suppose that the functor F is normal. In this situation \sim^F is an equivalence relation and coincides with $r_{\ker_F}^{\mathcal{C}, l}$*

Proof. Suppose that F is normal. Then $A = R(1)$ is self trivializing by Proposition 5.7 of [2]. Therefore $R(1)^{\otimes n} = [\text{FPdim}(R(1))]^n R(1)$. Since $R(1)^* = R(1)$ Example 2.7 implies that \sim^F is the equivalence relation $r_A^{\mathcal{C}, r}$. On the other hand by Proposition 5.7 of [2] it follows that the fusion subcategory $\langle R(1) \rangle$ coincides with \ker_F . \square

4.1. Description of the image of a normal tensor functor. Let as before $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_l$ be the equivalence classes of $r_{\ker_F}^{\mathcal{C}, l}$ on $\Lambda_{\mathcal{C}}$ and let

$$(4.1) \quad A_i = \sum_{X \in \mathcal{C}_i} \text{FPdim}(X)[X]$$

for $1 \leq i \leq l$.

Theorem 4.4. *Suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a normal tensor functor with a right adjoint R .*

1) *If $X, X' \in \Lambda_{\mathcal{C}}$ then $X \approx^F X'$ if and only if*

$$\frac{F(X)}{\text{FPdim}(X)} = \frac{F(X')}{\text{FPdim}(X')}.$$

2) *If $Y, Y' \in \Lambda_{\mathcal{D}}$ then $Y \approx_F Y'$ if and only if*

$$\frac{R(Y)}{\text{FPdim}(Y)} = \frac{R(Y')}{\text{FPdim}(Y')}.$$

Proof. 1) Suppose that \mathcal{C}_1 is the equivalence class of the unit object. Then clearly by definition $\mathcal{C}_1 = \ker_F$ and $A = R(1)$ is the regular object of \mathcal{C}_1 . If $X \in \mathcal{C}_i$ then relation 2.3 implies that

$$A \otimes X \cong \frac{\text{FPdim}(A)\text{FPdim}(X)}{\text{FPdim}(A_i)} A_i$$

and therefore

$$F(X) = \frac{\text{FPdim}(A)\text{FPdim}(X)}{\text{FPdim}(A_i)} F(A_i).$$

This shows that $X \approx_F X'$ if and only if

$$\frac{F(X)}{\text{FPdim}(X)} = \frac{F(X')}{\text{FPdim}(X')}.$$

2) Suppose that $X \in \mathcal{C}_i$. Then clearly $F(X)$ has all the constituents inside \mathcal{D}_i . Thus one has:

$$\begin{aligned} R(Y) &= \sum_{X \in \Lambda_{\mathcal{C}}} m_{\mathcal{D}}(R(Y), X) X = \sum_{X \in \Lambda_{\mathcal{C}}} m_{\mathcal{D}}(Y, F(X)) X \\ &= \sum_{X \in \mathcal{A}_i} m_{\mathcal{D}}(Y, F(X)) X \\ &= \sum_{X \in \mathcal{A}_i} m_{\mathcal{D}}(Y, \frac{F(X)}{\text{FPdim}(X)}) \text{FPdim}(X) X \\ &= m_{\mathcal{D}}(Y, \frac{F(A_i)}{\text{FPdim}(A_i)}) \sum_{X \in \mathcal{A}_i} \text{FPdim}(X) X \\ &= \text{FPdim}(Y) \frac{A_i}{\text{FPdim}(A_i)}. \end{aligned}$$

□

Proposition 4.5. *Suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a tensor functor and let \mathcal{D}' be the dominant image of F . The following assertions are equivalent:*

- (1) F is normal.
- (2) for $X, X' \in \Lambda_{\mathcal{C}}$, X^F and X'^F are either disjoint or equal (so they are the equivalence classes of an equivalence relation \sim_F on $\Lambda'_{\mathcal{D}}$);
- (3) for $Y, Y' \in \Lambda_{\mathcal{D}}$, Y_F and Y'_F are either disjoint or equal (so they form the classes of an equivalence relation \sim^F on $\Lambda_{\mathcal{C}}$); disjoint or equal;

Proof. (2) \implies (1). Assume $1 \in X^F$. We have also $1 \in 1^F$, so $X^F = 1^F$, that is $F(X)$ is trivial.

(1) \implies (2). Assume $Y \in X^F \cap X'^F$. Then $F(X' \otimes X^*)$ contains 1 so there exists $S \subset X' \otimes X^*$ simple, such that $1 \in S^F$. Then $F(S) = 1^n$. We have $X' \subset S \otimes X$ so $F(X') \subset F(X)^n$, hence $X'^F \subset X^F$. By reason of symmetry, $X'^F = X^F$.

(2) \implies (3). Assume $Y_F \cap Y'_F \neq \emptyset$. Let $X \in Y_F \cap Y'_F$ and $X' \in Y_F$. We must show $X' \in Y'^F$. Now $Y \in X^F$, $Y' \in X^F$ and $Y \in X'^F$. So $X'^F = X^F$ and $Y' \in X'^F$, that is $X' \in Y'^F$.

(3) \implies (2) is similar. \square

The above Proposition can be regarded as the analogue fact that a Hopf subalgebra is depth two if and only if it is normal. See [4] for a proof in the context of Hopf algebras.

As a consequence of that, if F is normal then F induces a bijection $\Lambda_C / \simeq^F \xrightarrow{\sim} \Lambda_{\mathcal{D}'} / \simeq_F$.

4.1.1. *Costes for normal fusion subcategories.* Recall [2] that a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is called normal if there is a normal tensor functor $F : \mathcal{C} \rightarrow \mathcal{E}$ such that $\mathcal{D} = \ker F$.

The following Proposition can be seen as a generalization of the fact that left and right cosets of a normal subgroup coincide. Its analogue for Hopf algebras was proven in [3].

Proposition 4.6. *If \mathcal{D} is a normal fusion subcategory of \mathcal{C} then $r_{\mathcal{D}}^{\mathcal{C}, l} = r_{\mathcal{D}}^{\mathcal{C}, r}$. Therefore the left and right cosets of \mathcal{C} relative to \mathcal{D} coincide.*

Proof. Since \mathcal{D} is normal there is an exact sequence

$$\mathcal{D} \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{E}$$

with $\mathcal{D} = \ker_F$. Then one can apply Theorem 4.3 and Lemma 3.3. \square

Remark 4.7. Then F is dominant $\iff \Lambda_{\mathcal{C}}^F = \Lambda_{\mathcal{D}}$, and F is normal $\iff ((1)_F)^F = \{1\}$.

Proposition 4.8. *If \mathcal{D} normal then $r_{\mathcal{D}}$ is central in $K_0(\mathcal{C})$.*

4.2. Functors from finite group actions on fusion categories.

Consider an action by tensor autoequivalences $\rho : G \rightarrow \text{Aut}_{\otimes}(\mathcal{C})$ of a finite group G on a fusion category \mathcal{C} . Let \mathcal{C}^G be the equivariantized fusion category (see [9] for its definition) and let $F : \mathcal{C}^G \rightarrow \mathcal{C}$ be the forgetful functor. Then it is a well known fact that F is a normal tensor functor.

Recall the following description of the simple objects of \mathcal{C}^G given in [5]. For any simple object $Y \in \Lambda_{\mathcal{C}}$ define the inertia subgroup:

$$(4.2) \quad G_Y = \{g \in G \mid \rho^g(Y) \cong Y\}.$$

By definition of G_Y , there exist isomorphisms $c^g : \rho^g(Y) \rightarrow Y$, for all $g \in G_Y$. For all $g, h \in G_Y$, the composition $c^g \rho^g(c^h)(\rho_{2_Y}^{g,h})^{-1}(c^{gh})^{-1}$ defines an isomorphism $Y \rightarrow Y$. Since Y is a simple object of \mathcal{C} , there exist nonzero scalars $\tilde{\alpha}_Y(g, h) \in k^*$ such that

$$(4.3) \quad \tilde{\alpha}_Y(g, h)^{-1} \text{id}_Y = c^g \rho^g(c^h)(\rho_{2_Y}^{g,h})^{-1}(c^{gh})^{-1} : Y \rightarrow Y.$$

This defines a map $\tilde{\alpha}_Y : G_Y \times G_Y \rightarrow k^*$ which is a 2-cocycle on G_Y . For any projective representation of G_Y with cocycle $\tilde{\alpha}_Y$ define

$$(4.4) \quad S_{Y,\pi} := \oplus_{t \in G/G_Y} \rho^t(Y) \otimes V_\pi$$

an object of \mathcal{C} with the following equivariant structure: for all $g \in G$, $\mu^g : \rho^g(S_{Y,\pi}) \rightarrow S_{Y,\pi}$ is defined componentwise by the formula

$$(4.5) \quad \mu^{g,t} = \rho^s(c^h)(\rho_2^{s,h})^{-1} \rho_2^{g,t} : \rho^g \rho^t(N) \rightarrow \rho^s(N),$$

where the elements $h \in G_Y$, $s \in G/G_Y$ are uniquely determined by the relation

$$(4.6) \quad gt = sh.$$

Recall from [5] that ρ_2 denotes the tensor structure of the tensor functor ρ . Thus $\rho_2^{g,h} : \rho^g \rho^h \rightarrow \rho^{gh}$ is a tensor isomorphism between the two tensor functors $\rho^g \rho^h$ and ρ^{gh} .

Let $H \subset G$ be any subgroup. Fix a simple object $Y \in \Lambda_{\mathcal{C}}$ and let $\mathcal{O}_G(Y)$ be the orbit of Y under the action of G . Thus $\mathcal{O}_G(Y) = \{\rho^g(Y), g \in G/G_Y\}$. Let also H_Y be the inertia subgroup of Y under the action of the subgroup H of G on \mathcal{C} . Then $H_Y = H \cap G_Y$ and the orbit $\mathcal{O}_G(Y)$ is a disjoint union of H orbits of some conjugated objects, say $\rho_1^g(Y), \dots, \rho_s^g(Y)$ where $\{g_i\}_{i=1}^s$ is a transversal of left cosets for G_Y/H_Y . Thus one can write that

$$\mathcal{O}_G(Y) = \bigsqcup_{i=1}^s \mathcal{O}_H(\rho^{g_i}(Y))$$

With the above notations then one has the following restriction formula:

$$(4.7) \quad \text{Res}_H^G(S_{Y,\pi}) = \oplus_{i=1}^s T_{\rho^{g_i}(Y), \pi \downarrow_{H_Y}^{G_Y}}$$

where $T_{\rho^{g_i}(Y), \pi \downarrow_{H_Y}^{G_Y}}$ are the corresponding simple objects of \mathcal{C}^H .

Proposition 4.9. *Let $\rho : G \rightarrow \text{Aut}_{\otimes}(\mathcal{C})$ be an action by tensor autoequivalences of a finite group G on a fusion category \mathcal{C} and H be a subgroup of G . Then the restriction functor $\text{Res}_H^G : \mathcal{C}^G \rightarrow \mathcal{C}^H$ is a normal tensor functor if and only if H is a normal subgroup of G .*

Proof. Note that $1_{\mathcal{C}^G} = S_{1, \epsilon_G}$ where ϵ_G is the trivial representation of $G_1 = G$. Similarly $1_{\mathcal{C}^H} = T_{1, \epsilon_H}$. For $Y = 1$, Formula 4.7 implies that $S_{1, \pi}$ viewed as an object of \mathcal{C}^H contains the unit $1_{\mathcal{C}^H}$ if and only if $\pi \downarrow_H^G$ contains the trivial representation of H . Thus Res_H^G is a normal tensor functor if and only if the functor $\text{res}_H^G : \text{Rep}(G) \rightarrow \text{Rep}(H)$ is a normal tensor functor. This is equivalent to the normality of the subgroup H in G . \square

5. ON THE RADICAL AND COMMUTATOR OF A NORMAL FUSION SUBCATEGORY

In this section we introduce the notion of a radical of a fusion subcategory and study its properties for normal fusion subcategories.

Lemma 5.1. *If \mathcal{D} is a normal fusion subcategory of \mathcal{C} then $\mathcal{D} \cap \mathcal{C}_1$ is a normal fusion subcategory of \mathcal{C}_1 for any fusion subcategory \mathcal{C}_1 of \mathcal{C} .*

5.1. On the commutator of a normal fusion subcategory.

5.1.1. *Definition of the commutator.* Recall the notion of commutator subcategory from [8]. If \mathcal{D} is a fusion subcategory of \mathcal{C} then \mathcal{D}^{co} is the full abelian subcategory of \mathcal{C} generated by those simple objects X such that $X \otimes X^* \in \mathcal{O}(\mathcal{D})$. If $K_0(\mathcal{C})$ is commutative then \mathcal{D}^{co} is a fusion subcategory of \mathcal{C} (see [8]).

Lemma 5.2. *Suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a tensor functor and $X, Y \in \mathcal{O}(\mathcal{C})$. Then one has that $X \otimes Y \in \ker_F$ if and only if*

$$(5.1) \quad \frac{F(X)}{\text{FPdim}(X)} = \frac{F(Y^*)}{\text{FPdim}(Y^*)} = M$$

for some $M \in \text{Inv}(\mathcal{C})$, the set of all invertible objects of \mathcal{C} .

Proof. Suppose that $F(X) = \sum_{Z \in \text{Irr}(\mathcal{C})} m_Z Z$ and $F(Y) = \sum_{Z \in \text{Irr}(\mathcal{C})} n_Z Z$ for some scalars $m_Z, n_Z \in k$. Then

$$(5.2) \quad F(X \otimes Y) \cong F(X) \otimes F(Y) = \sum_{Z, Z' \in \text{Irr}(\mathcal{C})} m_Z n'_Z (Z \otimes Z')$$

It follows that if $m_Z n_{Z'} \neq 0$ then $1_{\mathcal{C}}$ is a constituent of $Z \otimes Z'$ and therefore $Z' = Z^*$. On the other hand since the multiplicity of $1_{\mathcal{C}}$ in $Z \otimes Z^*$ is one it follows that $Z \otimes Z^* = 1_{\mathcal{C}}$ and therefore Z is an invertible object of \mathcal{C} . \square

5.2. Definition of the radical of fusion subcategory. For a fusion subcategory \mathcal{D} of \mathcal{C} define its radical as

$$(5.3) \quad \text{rad}_{\mathcal{C}}(\mathcal{D}) = \{X \in \text{Irr}(\mathcal{C}) \mid X^{\otimes n} \in \mathcal{D}\}$$

If $K_0(\mathcal{C})$ is commutative (for example \mathcal{C} braided) then clearly $\text{rad}_{\mathcal{C}}(\mathcal{D})$ is a fusion subcategory.

5.2.1. The commutator and the radical of a fusion subcategory. We show that for normal fusion subcategories the radical and the commutator coincide.

Proposition 5.3. *Suppose the \mathcal{D} is a normal fusion subcategory of \mathcal{C} fitting into the exact sequence*

$$(5.4) \quad \mathcal{D} \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{E}$$

Then \mathcal{D}^{co} and $\text{rad}_{\mathcal{C}}(\mathcal{D})$ are fusion subcategories of \mathcal{C} . Moreover:

$$\mathcal{D}^{\text{co}} = \text{rad}_{\mathcal{C}}(\mathcal{D}) = \{X \in \mathcal{C} \mid F(X) = \text{FPdim}(X)M, M \in \text{Inv}(\mathcal{E})\}$$

Proof. First we show that

$$\mathcal{D}^{\text{co}} = \{X \in \mathcal{C} \mid F(X) = \text{FPdim}(X)M, M \in \text{Inv}(\mathcal{E})\}.$$

If $X \otimes X^* \in \mathcal{D}$ then by Lemma 5.2 it follows that $F(X) = \text{FPdim}(X)M$ for some, $M \in \text{Inv}(\mathcal{E})$. Thus $\mathcal{D}^{\text{co}} \subseteq \{X \in \mathcal{C} \mid F(X) = \text{FPdim}(X)M, M \in \text{Inv}(\mathcal{E})\}$. For the other inclusion, if $F(X) = \text{FPdim}(X)M$ for some, $M \in \text{Inv}(\mathcal{E})$ then $F(X \otimes X^*) = \text{FPdim}(X \otimes X^*)1_{\mathcal{E}}$ and therefore $XX^* \in \mathcal{D}$.

Next we show that

$$\text{rad}_{\mathcal{C}}(\mathcal{D}) = \{X \in \mathcal{C} \mid F(X) = \text{FPdim}(X)M, M \in \text{Inv}(\mathcal{E})\}$$

Suppose that $F(X) = \text{FPdim}(X)M$ for some, $M \in \text{Inv}(\mathcal{E})$. It follows that there is $n \geq 1$ such that $M^{\otimes n} = \text{FPdim}(M)^n 1_{\mathcal{E}}$ since the group of invertible objects $\text{Inv}(\mathcal{E})$ is finite. This implies that $F(X^{\otimes n}) = \text{FPdim}(X)^n 1_{\mathcal{E}}$ which implies that $X^{\otimes n} \in \mathcal{D}$, i.e $X \in \text{rad}_{\mathcal{C}}(\mathcal{D})$. Conversely, if $X \in \text{rad}_{\mathcal{C}}(\mathcal{D})$ then $F(X) = \text{FPdim}(X)M$, for some $M \in \text{Inv}(\mathcal{E})$ by Lemma 5.2. \square

5.3. Group extensions of normal fusion subcategories.

Proposition 5.4. *Suppose that \mathcal{C} is a graded fusion category*

$$(5.5) \quad \mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

with the grading component \mathcal{C}_1 a normal fusion subcategory of \mathcal{C} . Then each \mathcal{C}_g is a double coset relative to \mathcal{C}_1 , i.e $\mathcal{C}_g = \mathcal{C}_1 X_g = X_g \mathcal{C}_1$ for any element $X_g \in \mathcal{C}_g$.

Proof. Since \mathcal{C}_1 is a normal fusion subcategory \mathcal{C} it follows by Proposition 4.6 that left and right cosets of \mathcal{C}_1 inside \mathcal{C} coincide. Thus $\mathcal{C}_1 X_g = X_g \mathcal{C}_1$. On the other hand from the grading 5.5 clearly $\mathcal{C}_1 X_g \subseteq \mathcal{C}_g$. It remains to show the other inclusion. Suppose that \mathcal{C}_1 fits into the exact sequence:

$$(5.6) \quad \mathcal{C}_1 \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{E}$$

for some fusion category \mathcal{E} . Let $X \in \mathcal{C}_g$ an arbitrary simple object. Since $g^n = 1$ for $n = o(g)$, the order of g , it follows that $X^{\otimes n} \in \mathcal{C}_1$. Then by Proposition 5.3 one has that $X \in \mathcal{C}_1^{co}$ and therefore $F(X) = \text{FPdim}(X)M$ for some invertible $M \in \text{Inv}(\mathcal{E})$. Similarly it follows that $F(X_g) = \text{FPdim}(X_g)M_g$ for some other $M_g \in \text{Inv}(\mathcal{E})$. Therefore $F(X^* X_g) = \text{FPdim}(X^* X_g)M^* M_g$. But the grading 5.5 implies that $X^* X_g \in \mathcal{C}_1$ which in turn implies that $F(X^* X_g) = \text{FPdim}(X^* X_g)1_{\mathcal{E}}$. Thus $M = M_g$ and Proposition 4.4 applied to the tensor functor F implies that $X \in \mathcal{C}_1 X_g$. Therefore $\mathcal{C}_g = \mathcal{C}_1 X_g = \mathcal{C}_1 X_g$. \square

Corollary 5.5. *Let \mathcal{C} be a normal G -extension of \mathcal{D} . Then there is a one to one map between the set of all fusion subcategories of \mathcal{C} containing \mathcal{D} and the set of all subgroups of G .*

Proof. For a fusion subcategory $\mathcal{D} \subset \mathcal{E} \subset \mathcal{C}$ define

$$(5.7) \quad H_{\mathcal{E}} := \{g \in G \mid \mathcal{E} \cap \mathcal{C}_g \neq 0\}$$

Then it is easy to check that $H_{\mathcal{E}}$ is a subgroup of G and $\mathcal{E} \mapsto H_{\mathcal{E}}$ is a bijective map from the set of all fusion subcategories of \mathcal{C} containing \mathcal{D} to the set of all subgroups of G . \square

Remark 5.6. Suppose that \mathcal{C} is a normal G -extension of a fusion subcategory \mathcal{D} . The proof of the previous Proposition also shows that $\mathcal{D}^{co} = \mathcal{C}$. Examples of normal G -extension are given by $\text{Rep}(A//K) \subset \text{Rep}(A)$, where K is a central Hopf subalgebra of A .

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